TIGHT HAMILTON CYCLES IN CHERRY QUASIRANDOM 3-UNIFORM HYPERGRAPHS

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ABSTRACT

We employ the absorbing-path method in order to prove that for any fixed real \( \alpha > 0 \) the so called two-path or cherry quasirandom 3-uniform hypergraphs of sufficiently large order \( n \) having minimum 1-degree at least \( \alpha \left( \frac{n-1}{2} \right) \) admit a tight Hamilton cycle.

§1. Introduction

A theorem of Dirac [10] asserts that an \( n \)-vertex (\( n \geq 3 \)) graph whose minimum degree is at least \( n/2 \) contains a Hamilton cycle; moreover, the degree condition imposed here is best possible. A rich and extensive body of work now exists concerning the extent to which Dirac’s result can be extended to uniform hypergraphs see, e.g., [3, 7, 8, 12, 13, 15, 16, 18, 19, 21, 22, 32, 33, 34, 35, 37]\(^1\). Allow us to not reproduce here the intricate development of these results as outstanding accounts of these already exist in the excellent surveys [23, 31, 43].

We confine ourselves to 3-uniform hypergraphs (3-graphs, hereafter). A 3-graph \( C \) is said to be a loose cycle if its vertices can be cyclically ordered such that each edge of \( C \) captures 3 vertices appearing consecutively in the ordering, every vertex is contained in an edge, and any two consecutive edges meet in precisely one vertex. We say \( C \) is a tight cycle if there exists a cyclic ordering of its vertices such that every 3 consecutive vertices in this ordering define an edge of \( C \); this particularly implies that any two consecutive edges meet in precisely 2 vertices.

For a 3-graph \( H \) and two distinct vertices \( u \) and \( v \) of it define

\[
\text{deg}_H(v) := |N_H(v)| := |\{x, y \in \binom{V(H)}{2} : \{x, y, v\} \in E(H)\}| = |\{e \in E(H) : v \in e\}|
\]

\[
\text{deg}_H(u, v) := |N_H(u, v)| := |\{w \in V(H) : \{u, v, w\} \in E(H)\}| = |\{e \in E(H) : \{u, v\} \subset e\}|
\]

We refer to \( \text{deg}_H(v) \) as the degree of \( v \) (alternatively, 1-degree) and to \( \text{deg}_H(u, v) \) as the codegree of \( u \) and \( v \) (alternatively, 2-degree). Set

\[
\delta(H) := \min_{v \in V(H)} \text{deg}_H(v) \text{ and } \delta_2(H) := \min_{\{u, v\} \in \binom{V(H)}{2}} \text{deg}_H(u, v).
\]

Resolving a conjecture of [18] first approximately [34] and then accurately [37], the latter result asserts that a sufficiently large \( n \)-vertex 3-graph \( H \) satisfying \( \delta_2(H) \geq \lceil n/2 \rceil \) contains a tight Hamilton cycle. A construction appearing in [18] demonstrates that the Dirac-type condition imposed

\(^1\)The study of perfect matchings in hypergraphs is intimately related to the Hamiltonicity problem. We omit references to such results as our work here was not directly influenced by this line of research.

\(^2\)Order of the edges inherited from the ordering of the vertices.
here is best possible. Finding the correct threshold for $\delta(H)$ at which a 3-graph $H$ admits a tight Hamilton cycle remained elusive for quite some time. The problem has come to be known as the $5/9$-conjecture [31, Conjecture 2.18] asserting that a sufficiently large $n$-vertex 3-graphs $H$ satisfying $\delta(H) \geq (5/9 + o(1))(n-2)$ admit a tight Hamilton cycle. Constructions appearing in [31, 32] establish that the conjecture, if correct, then its Dirac-type condition is (asymptotically) best possible. The authors of [6] established that such 3-graphs admit a tight cycle covering all but $o(n)$ of the vertices. Then, in a major breakthrough [27] (preceded by the deep result of [33] and around the same time as [6]), the $5/9$-conjecture has been resolved.

The last result relevant to us is that of [26]; presentation of which requires a brief overview regarding quasirandom 3-graphs. Launched in [4, 40, 41], the study of quasirandom graphs has developed into a rich and vast theory, see, e.g. [20]. While a canonical definition of quasirandom graphs was already captured in [4, 40, 41], for hypergraphs the pursuit after a definition extending [4] took much longer. An elaborate account regarding the development of this pursuit can be seen in [1, 5, 24, 25, 42] and references therein. Only recently with the work of [42] has this pursuit came to an end; an alternative combinatorial approach to the functional analytic work of [42] appears in [1].

Roughly speaking, for $k \geq 3$ each set system of $[k] = \{1, \ldots, k\}$ forming a maximal anti-chain gives rise to a notion of quasirandomness for $k$-graphs. In the case of interest to us, that is $k = 3$, each of the maximal anti-chains

$$\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 2\}, \{2, 3\}\}, \text{and } \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$$

defines a notion of quasirandomness referred to as $\bullet$-quasirandomness with $\bullet \in \{\star, \Delta\}$, respectively (concrete definitions follow below); here these notions are arranged from left to right in increasing order of strength sort of speak.

A solid understanding of $\star$-quasirandomness (i.e., the weakest notion) was attained in [5, 24]. More generally, we now know from [1, 42] (and owing much to [25]) that all these notions are well-separated and form a certain hierarchy with $\star$-quasirandomness at the "bottom" as the weakest notion (so it forms the broadest class of hypergraphs). In what follows, however, we will not be bothered with these notions of quasirandomness per say. Instead we shall consider weaker related notions. Borrowing notation from [28, 29], given $d, \varrho \in (0, 1]$, an $n$-vertex 3-graph $H$ is said to be $(\varrho, d)\Delta$-dense if

$$e_H(X, Y, Z) := |\{(x, y, z) \in X \times Y \times Z : \{x, y, z\} \in E(H)\}| \geq d|X||Y||Z| - \varrho n^3 \quad (1.1)$$

holds for every $X, Y, Z \subseteq V(H)$. If $\varrho$ and $d$ exist yet are not made explicit we say $H$ is $\bullet$-dense. The notion of $\star$-quasirandomness comes about if one imposes on $e_H(X, Y, Z)$ the upper bound corresponding to (1.1).

We return to Hamiltonicity and the following remarkable result of [26] stated here for 3-graphs only.

**Theorem 1.2.** [26] For every $d, \alpha \in (0, 1]$ there exist an $n_0$ and a $\varrho > 0$ such that the following holds whenever $n \geq n_0$ and even. Let $H$ be an $n$-vertex $(\varrho, d)\Delta$-dense 3-graph satisfying $\delta(H) \geq \alpha(n^{-1})$. Then $H$ admits a loose Hamilton cycle.

Theorem 1.2 settles the issue of emergence of loose Hamilton cycles in quasirandom 3-graphs for any notion of quasirandomness and any type of degree (the latter owing to [31, Remark 1.4]). It asserts
that all Dirac-type conditions sufficient for the emergence of loose Hamilton cycles in quasirandom 3-graphs are degenerate (i.e., any positive $\alpha$ suffices).

For tight cycles, however, a result analogous to Theorem 1.2 does not exist for -quasirandom 3-graphs. Indeed, [26, Proposition 4] asserts that for every $\varrho > 0$ and sufficiently large $n$ an $n$-vertex $(\varrho, 1/8)$-quasirandom 3-graph $H$ exists satisfying $\delta(H) \geq (1/8 - \varrho)^{n-1}$ and having no tight Hamilton cycle. The constant $1/8$ here is not best possible though as the following construction demonstrates. Let $n \in \mathbb{N}$ be sufficiently large and let $V = X \cup Y$ be a set of $n$ vertices such that $|X| = 2n/3 + 1$ and $|Y| = n/3 - 1$ (assume $3 \mid n$). Let $G \sim G(n, p)$ be the random graph put on $V$ where each edge is put in $G$ independently at random with probability $p$; we determine $p$ below. Define $H$ to be the 3-graph whose set of vertices is $V$ and whose set of edges consists of:

- all the sets $e \in \binom{V}{3}$ satisfying $G[e] \cong K_3$ and $e \subseteq X$ or $e \subseteq Y$ or $|e \cap X| = 1$;

- together with the sets $e \in \binom{V}{3}$ satisfying $2 = |e \cap X| := |\{u, v\}|$ and $uv \notin E(G)$.

An argument similar to the one used in [32, Construction 2] asserts that $H$ has no tight Hamilton cycle. Indeed, no tight path can connect a triple contained in $X$ with a vertex of $Y$. Consequently, if $H$ were to admit a tight Hamilton cycle $C$ then $X$ must be an independent set in $C$ and $Y$ a vertex-cover of $C$. This together with the fact that $C$ is 3-regular (with respect to 1-degree, that is) we reach $n = e(C) \leq \sum_{y \in Y} \deg_C(y) = 3|Y| < n$; a contradiction. Every triple $e$ is taken into $H$ either with probability $p^3$ or $1 - p$. Insisting on $p^3 = 1 - p$, so that $p = 0.68$. Using binomial tail estimations it follows that it is highly likely that $H$ would have edge density $\approx 0.314$, satisfy $\delta(H) \approx 0.245n^2$, and be -dense. We acknowledge the discussions [30] regarding this construction.

Replacing the degree condition seen in Theorem 1.2 with a codegree condition would be insufficient in order to yield a result analogous to Theorem 1.2. Indeed, in [26] it is indicated that an adaption of the construction [26, Proposition 4] yields a -dense graph $H$ with $\delta_2(H) \geq n/9$ admitting no tight Hamilton cycle.

§1.1 Our Result. If we were to "climb" up the hierarchy of notions of quasirandomness for 3-graphs and strengthen the quasirandomness condition satisfied by the host 3-graph would we then encounter an analogue of Theorem 1.2 for tight Hamilton cycles? Let $d, \varrho \in (0, 1]$. An $n$-vertex 3-graph $H$ is called $(\varrho, d)$- if

$$e_H(G_1, G_2) := |\{(x, y, z) \in \mathcal{P}_2(G_1, G_2) : (x, y, z) \in E(H)\}| \geq d|\mathcal{P}_2(G_1, G_2)| - gn^3$$

holds for every $G_1, G_2 \subseteq V(H) \times V(H)$, where

$$\mathcal{P}_2(G_1, G_2) := \{(x, y, z) \in V(H)^3 : (x, y) \in G_1, (y, z) \in G_2\}.$$ 

If $\varrho$ and $d$ exist yet are not made explicit we say that $H$ is (pronounced cherry-dense).

The following is our main result. It asserts that the Dirac-type conditions (for any type of degree) for tight Hamilton cycles in -dense 3-graphs are degenerate.

**Theorem 1.4.** For every $d, \alpha \in (0, 1]$ there exist an $n_0$ and $\varrho > 0$ such that the following holds for all $n \geq n_0$. Let $H$ be an $n$-vertex $(\varrho, d)$-3-graph satisfying $\delta(H) \geq \alpha(n^{-1})$. Then $H$ has a tight Hamilton cycle.

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For hypergraphs with higher uniformity the full version of Theorem 1.2 handles the emergence of the so called 1-cycles.
"Between" $\preceq$-quasirandomness and $\succeq$-quasirandomness there lies $\succeq$-quasirandomness. For $d, \varrho \in (0,1)$ an $n$-vertex 3-graph $H$ is called $(d, \varrho)$-$\succeq$-dense if

$$e_H(P, X) := |\{(u, v, x) \in P \times X : \{u, v, x\} \in E(H)\}| \geq d|P||X| - \varrho n^3$$

holds for every $P \subseteq V(H) \times V(H)$ and every $X \subseteq V(H)$. Unlike $\succeq$-quasirandom 3-graphs, for which the Turán density of $K_{4}^{(3)}$ (the complete 3-graph on 4 vertices) is zero [28], the Turán density of $K_{4}^{(3)-}$ (i.e., $K_{4}^{(3)}$ with a single edge removed) in $\succeq$-quasirandom 3-graphs is $1/4$ [29].

The absorbing configurations (see § 4 for details) used here involve copies of $K_{4}^{(3)-}$. Consequently results in the spirit of Theorem 1.4 cannot possibly be attained for $\succeq$-quasirandom 3-graphs using the absorbing-path method and the absorbing configurations used here. We subscribe to the point of view that the flaw is not in the method and that for $\succeq$-quasirandom 3-graphs the Dirac-type conditions for 1-degree and 2-degree implying tight Hamiltonicity are both non-degenerate. The fact that the Turán density of $K_{4}^{(3)-}$ in $\succeq$-quasirandom 3-graphs coincides with that seen in $\succeq$-quasirandom 3-graphs [29] makes it not far-fetched to suspect that the Dirac-type conditions in $\{\preceq, \succeq\}$-quasirandom 3-graphs coincide as well. The construction above suggests a place to start this investigation.

§1.2 Our approach. We employ the so called absorbing path method introduced in [34] and further developed in [35, 36]. Roughly speaking, this method reduces the problem of finding a (tight) Hamilton cycle to that of finding a (tight) cycle supporting two properties. First, it covers all but $\zeta n$ vertices for some carefully chosen fixed (and "small") $\zeta \in (0,1)$; second, it contains a special path referred to as an absorbing-path (rigorously defined below) which has the capability of being rerouted using only those "missing" $\zeta n$ vertices while keeping its ends unchanged and in this manner absorb, sort of speak, all missing vertices rendering a (tight) Hamilton cycle. Many reincarnations of this method now exist in the literature see e.g., [13, 26, 27, 33]. We consequently omit a more rigorous outline of this method and proceed directly to the statement of the so called pillar lemmata of this method; these being the so called connecting lemma, absorbing-path lemma, path-cover lemma; the reservoir lemma (i.e., Lemma 6.1) will be stated later on in § 6.

By $k$-path we mean a 3-graph $P$ on $k$ vertices and $k-2$ edges such that there exists a labelling of $V(P)$ namely $v_1, \ldots, v_k$ such that $\{v_i, v_{i+1}, v_{i+2}\} \in E(P)$ for every $i \in [1, k-2]$. It is said that $P$ connects the pairs $\{v_1, v_2\}$ and $\{v_{k-1}, v_{k-2}\}$; also referred to as the end-pairs or simply the ends of $P$. Throughout the term path is used to denote a tight path.

Roughly speaking, the connecting lemma is in charge of connecting two disjoint pairs of vertices via a short path. A trivial precondition for doing so is that the given pairs admit some non-trivial codegree. A useful minimum codegree condition for the 3-graphs of Theorem 1.4 cannot be inferred. Nevertheless, these 3-graphs come equipped with a certain "statistical" minimum codegree condition (see § 2 for details). Hence the following.

Let $H$ be a 3-graph. For a real $\beta > 0$ let $H_\beta$ denote the 3-graph obtained from $H$ by removing all edges of $H$ containing a pair whose codegree is $< \beta |V(H)|$. For a $\succeq$-dense $H$ the non-triviality of $H_\beta$ is captured through (2.5) below. The pairs of vertices contained in at least one edge of $H_\beta$ we call $\beta$-relevant. The following lemma asserts that for an appropriate choice of $\beta$, $\beta$-relevant pairs can be connected through a 'short' path in $H$ (i.e., the connecting path may involve edges not in $H_\beta$). The following lemma imposes no Dirac-type conditions.

Lemma 1.5. (Connecting lemma) For every $d_{1.5}, \beta_{1.5} \in (0,1]$ such that $\beta_{1.5} < d_{1.5}$ there exist
an integer \( n_{1.5} \) and a real \( q_{1.5} := q_{1.5}(d_{1.5}, \beta_{1.5}) > 0 \) such that the following holds for all \( n \geq n_{1.5} \) and \( 0 < q < q_{1.5} \). Let \( H \) be an \( n \)-vertex \((q, d_{1.5})\)\(\clubsuit\)-dense 3-graph and let \( \{x,y\} \) and \( \{x',y'\} \) be two disjoint \( \beta \)-relevant pairs of vertices of \( H \). Then there exists a 10-path in \( H \) connecting \( \{x,y\} \) and \( \{x',y'\} \).

A path \( A \) in a 3-graph \( H \) is said to be \( m \)-absorbing if for every set \( U \subseteq V(H) \setminus V(A) \) with \( |U| \leq m \) there is a path \( A_U \) having the same ends as \( A \) and satisfying \( V(A_U) = V(A) \cup U \). In view of our connecting lemma we shall require the end pairs of such a path to be \( \beta \)-relevant for some properly chosen \( \beta > 0 \). The following is the sole pillar lemma imposing a Dirac-type condition.

**Lemma 1.6. (Absorbing-path lemma)** For every \( d_{1.6}, \alpha_{1.6}, \beta_{1.6} \in (0,1] \) such that \( d_{1.6}^2 \alpha_{1.6}^9 < \beta_{1.6} \) there exist an integer \( n_{1.6} \), a real \( q_{1.6} := q_{1.6}(d_{1.6}, \alpha_{1.6}, \beta_{1.6}) > 0 \), a real \( 0 < \kappa_{1.6} := \kappa_{1.6}(d_{1.6}, \alpha_{1.6}) < d_{1.6}^3 \alpha_{1.6}^9/10 \), and a real \( m_{1.6} := m_{1.6}(d_{1.6}, \alpha_{1.6}) \in (0,1] \) such that the following holds whenever \( n \geq n_{1.6} \) and \( 0 < q < q_{1.6} \). Let \( H \) be an \( n \)-vertex \((q, d_{1.6})\)\(\clubsuit\)-dense 3-graph satisfying \( \delta(H) \geq \alpha_{1.6}(n-1)/2 \). Then there exists an \( m_{1.6} \)-absorbing \( \kappa_{1.6} \) path \( A \) in \( H \) whose end pairs are both \( \beta \)-relevant.

For the next pillar lemma \( \heartsuit \)-denseness is not required. Here a notion weaker from \( \heartsuit \)-dense will suffice; where the latter notion is the one essentially used in [35, Section 4]. Let \( d, q \in (0,1] \) and let \( H \) be an \( n \)-vertex 3-graph. If

\[
eq_{H}(X) := \size{E(H) \cap \binom{X}{3}} \geq d \binom{|X|}{3} - qn^3 \tag{1.7}\]

holds for every \( X \subseteq V(H) \) we say that \( H \) is \((q, d)\)-dense. If \( q \) and \( d \) are known to exist yet are not made explicit we say \( H \) is 1-set-dense\(^4\). The following lemma imposes no Dirac-type condition on the 3-graph as well.

**Lemma 1.8. (Path-cover lemma)** For every \( d_{1.8}, \zeta_{1.8} \in (0,1] \) there exist \( n_{1.8} \), \( q_{1.8} = q_{1.8}(d_{1.8}, \zeta_{1.8}) > 0 \), and an integer \( \gamma_{1.8} = \gamma_{1.8}(d_{1.8}, \zeta_{1.8}) \) such that the following holds for all \( n \geq n_{1.8} \) and \( 0 < q < q_{1.8} \). Let \( H \) be an \( n \)-vertex \((q, d_{1.8})\)-dense 3-graph. Then all but at most \( \zeta_{1.8}n \) vertices of \( H \) can be covered using at most \( \gamma_{1.8} \) vertex-disjoint paths.

\[\mathsection 2.\ \text{Pairs with positive codegree}\]

Let \( H \) be a 3-graph, let \( G \subseteq V(H) \times V(H) \), and let \( (u,v) \in V(H) \times V(H) \); we write

\[
\deg_H(u,v,G) := |\{ z \in V(H) : ((z,u) \in G \text{ or } (u,z) \in G) \text{ and } z,u,v \in E(H) \}|.
\]

Note that \( \deg_H(u,v,G) \neq \deg_H(v,u,G) \) is possible. In this definition \( G \) is treated as an undirected graph and indeed in the sequel we shall also write \( \deg_H(u,v,G) \) when \( G \) is an undirected graph. We allow this leniency since this definition will only be used when upper bounds are involved; for lower bounds we will appeal to \( \heartsuit \)-denseness. An ordered set of pairs is said to be non-degenerate if it contains no pairs of the form \((x,x)\). We find it more convenient to have the following lemma formulated using undirected graphs.

**Lemma 2.1.** Let \( d, \alpha, \text{ and } q \) be positive reals and let \( H \) be a \((q, d)\)\(\heartsuit\)-dense \(n\)-vertex 3-graph. Let

\[^4\text{If in addition to (1.7) } H \text{ also satisfies its corresponding upper bound then } H \text{ is } \heartsuit \text{-quasirandom (see e.g. [5]).}\]
\[ G \text{ be a graph on } V(H) \text{ with vertex cover } Y \subseteq V(G) \text{ such that} \]
\[ \deg_G(y) \geq k \quad \text{for all } y \in Y, \quad (2.2) \]
where \( k \) is an integer (which may depend on \( n \)). For an integer \( \Delta \) (which may depend on \( n \)) set
\[ B_\Delta := \{(y, z) \in Y \times V(H) : \deg_H(y, z, G) < \Delta\} \]
and suppose that \( B_\Delta \) is non-degenerate. Then \( |B_\Delta| \leq \frac{g n^3}{d k^{2\Delta}} \).

Proof. Owing to \( \beta \geq 1 \) and suppose that \( \beta \) is non-degenerate. Then
\[ \Delta = \{(y, z) : y \in Y, (y, z) \in B_\Delta\}. \]
For the upper bound observe that \( \beta \) extends into at most \( \sum_{y \in Y} \deg_G(y) \deg_{B_\Delta}(y) \geq k \sum_{y \in Y} \deg_{B_\Delta}(y) \geq k \cdot e(B_\Delta) \geq k|B_\Delta|/2; \)
for the third inequality we again rely on \( \beta \) being a vertex-cover of \( B_\Delta \) by definition; in the last inequality we use the assumption that \( B_\Delta \) is non-degenerate. Then
\[ dk|B_\Delta|/2 - gn^3 < 2|B_\Delta| \cdot \Delta; \]
The claim now follows upon isolating \( |B_\Delta| \) in the last inequality.

For an \( n \)-vertex 3-graph \( H \) and a real \( \beta > 0 \) recall \( H_\beta \) defined in § 1.2. Letting in addition, \( H \) be \((g, d)_\ast\)-dense and setting
\[ B_\beta := B_\beta(H) := \left\{ \{u, v\} \in \binom{V(H)}{2} : \deg_H(u, v) < \beta n \right\} \quad (2.3) \]
then a close variant of the argument seen in the proof of Lemma 2.1 a kin to taking in that lemma \( G \) to be the complete graph on \( V(H), Y = V(H), k = n, \) and \( \Delta = \beta n \) albeit with slightly different constants, yields
\[ d|P_2(V(H) \times V(H), \tilde{B})| - gn^3 \leq e_H(V(H) \times V(H), \tilde{B}) \leq |\tilde{B}| \beta n \]
where \( \tilde{B} := \{(u, v), (v, u) \in V(H) \times V(H) : \{u, v\} \in B_\beta\}; \) factor of 2 in the upper bound is no longer needed as here for every member of \( \tilde{B} \) we allow all possible extensions regardless of order. Then
\[ |B_\beta| \leq |\tilde{B}| \leq \frac{g}{d - \beta^2} n^2 \quad (2.4) \]
In particular, the non-triviality of $H_\beta$ is seen through
\[
e(H_\beta) \geq e(H) - \frac{\varrho}{d - \beta} n^3 \geq d \binom{n}{3} - \left(1 + \frac{1}{d - \beta}\right) \varrho n^3,
\]
where the last inequality is owing to the fact that $H$ is (trivially also) $(\varrho, d)$-dense. This means that for a sufficiently small $\varrho$ the 3-graph $H_\beta$ will indeed be non-trivial as long as $\beta < d$. The conditions $\beta_{1.5} < d_{1.5}$ and $\beta_{1.6} < d_{1.6}$ appearing in Lemmas 1.5 and 1.6, respectively, are imposed due to this issue.

Let $H$ be an $n$-vertex 3-graph. Set
\[
\delta^*_\beta(H) := \min\{\deg_H(u, v) : u, v \in V(H) \text{ and } \exists e \in E(H) \text{ s.t. } \{u, v\} \subseteq e\}.
\]
If $H_\beta$ is non-trivial then $\delta^*_\beta(H_\beta) \geq \beta n$, by definition. Throughout whenever we approach $H_\beta$ we always make sure to pick appropriate $\beta$ and $\varrho$ as to render $H_\beta$ non-trivial.

**Observation 2.6.** Let $H$ be an $n$-vertex 3-graph, let $\beta > 0$, and let $\{x, y\}$ be a $\beta$-relevant pair of $H$. If $z \in N_{H_\beta}(x, y)$ then $\{x, z\}$ and $\{y, z\}$ are $\beta$-relevant pairs of $H$ as well.

**Observation 2.7.** For every $0 < \kappa < \beta \leq 1$ there exists an $n_0$ such that for every $n \geq n_0$ the following holds. Let $H$ be an $n$-vertex 3-graph with $H_\beta$ non-trivial (i.e., $\delta^*_\beta(H_\beta) \geq \beta n$). If $\{x, y\}$ is $\beta$-relevant in $H$ then $\{x, y\}$ is $(\beta - \kappa)$-relevant in $H - U$ for any $U \subseteq V(H)$ satisfying $|U| \leq \kappa n$ and $\{x, y\} \cap U = \emptyset$.

**Proof.** Owing to $\delta^*_\beta(H_\beta) \geq \beta n$, the assumption that $\{x, y\}$ is $\beta$-relevant and $|U| < \beta n$ imply that for $n$ sufficiently large there exists an edge $\{x, y, z\} \in E(H_\beta)$ satisfying $z \notin U$. For this edge we note that
\[
\deg_{H - U}(x, y), \deg_{H - U}(x, z), \deg_{H - U}(y, z) \geq \beta n - |U| \geq (\beta - \kappa)n \geq (\beta - \kappa)|V(H - U)|.
\]
Hence $\{x, y, z\} \in E((H - U)_{\beta - \kappa})$ implying that $\{x, y\}$ is $(\beta - \kappa)$-relevant in $H - U$.

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**§3. Connecting Lemma**

**Cascades.** Let $n$ be a sufficiently large integer and let $H$ be an $n$-vertex 3-graph satisfying $\delta^*_\beta(H) \geq \beta n$ for some fixed real $\beta \in (0, 1]$ independent of $n$. Fix $x$ and $y$ to be two vertices in $H$ contained in an edge of $H$. Below we define a tuple
\[
\mathcal{C}(x, y) := \{x, y, N_1(x, y), N_2(x, y), N_3(x, y), G_1(x, y), G_2(x, y), G_3(x, y)\}
\]
and refer to it as an $\{x, y\}$-cascade. This definition we borrow from [34] and adapt it to fit our degenerate degree condition; this following the observation that to build cascades a minimum codegree condition as seen in [34] is not required; a condition on $\delta^*_\beta(H)$ suffices. In broad terms, for every $i \in [3]$, $N_i(x, y)$ denotes a set of vertices which, roughly speaking, plays the rôle of an $i$th coneighbourhood of the pair $\{x, y\}$. The parameters $(G_i(x, y))_{i \in [3]}$ represent certain graphs between these coneighbourhoods which will facilitate the tracking of 5-paths from $N_3(x, y)$ all the way (back) to $\{x, y\}$.
Let \( N_1 := N_1(x, y) := N_H(x, y) \). The assumption \( \delta_2^+(H) \geq \beta n \) implies
\[
|N_1| \geq \beta n. \tag{3.1}
\]

Define \( G_1 := G_1(x, y) \) to be the (bipartite) graph whose vertex set is \( \{y\} \cup N_1 \) and whose edges are given by the set \( \{yz : z \in N_1\} \). To define \( N_2 := N_2(x, y) \) and \( G_2 := G_2(x, y) \) set first
\[
N'_2 := N'_2(x, y) := \bigcup_{z \in N_1} N_H(y, z) = \{w \in V(H) : \exists z \in N_1 \text{ s.t. } \{y, z, w\} \in E(H)\}.
\]

Define \( G'_2 := G'_2(x, y) \) to be the graph whose vertex set is \( N_1 \cup N'_2 \) and whose edges are given by the set
\[
E(G_2) := \{zz' : z \in N_1, z' \in N'_2 \cap N_H(y, z)\} = \{zz' : z \in N_1, z' \in V(H), \text{ and } \{y, z, z'\} \in E(H)\},
\]
where we ignore loops if any are created so that \( G'_2 \) is simple. The assumption that \( \delta_2^+(H) \geq \beta n \) implies that
\[
\deg_{G'_2}(u) \geq \beta n \text{ for every } u \in N_1 \tag{3.2}
\]
and then
\[
e(G'_2) \geq \frac{1}{2} \sum_{u \in N_1} \deg_{G'_2}(u) \overset{(3.2)}{\geq} |N_1| \beta n / 2 \overset{(3.1)}{\geq} \beta^2 n^2 / 2. \tag{3.3}
\]

We shall have to discard members of \( N'_2 \) whose degree into \( N_1 \) is "too low" as follows. Set
\[
N_2^{(\text{low})} := N_2^{(\text{low})}(x, y) := \{z \in N'_2 : \deg_{G'_2}(z) < \log n\}.
\]
(The choice of \( \log n \) here is completely arbitrary. Any function \( \omega(n) \ll n \) growing slowly to \( \infty \) will suffice; this will become clear soon). Setting \( N_2 := N_2(x, y) := N_2 \setminus N_2^{(\text{low})} \) we arrive at
\[
\beta^2 n^2 / 2 \overset{(3.3)}{\leq} e(G'_2) \leq (\log n) \cdot |N_2^{(\text{low})}| + |N_2| \cdot |N_1| \leq n \log n + |N_2| \cdot n
\]
so that for \( n \) sufficiently large
\[
|N_2| \geq \beta^2 n / 4. \tag{3.4}
\]

Set \( G_2 := G_2(x, y) := G'_2[N_1 \cup N_2] \) and note that for \( n \) sufficiently large
\[
e(G_2) \geq e(G'_2) - n \log n \overset{(3.3)}{\geq} \beta^2 n^2 / 4. \tag{3.5}
\]

We turn to the definition of the set \( N_3 := N_3(x, y) \) and the graph \( G_3 := G_3(x, y) \). To that end associate an auxiliary graph \( B_w := B_w(x, y) \) with every vertex \( w \in N_2 \). In particular, for a fixed vertex \( w \in N_2 \) let \( B_w \) be the graph whose vertex set is \( V(H) \) and whose edges are given by the set
\[
E(B_w) := \{uz : u \in V(H), z \in N_{G_2}(w) \subseteq N_1, \text{ and } \{z, w, u\} \in E(H)\}.
\]

Using \( (B_w)_{w \in N_2} \) define
\[
N_3 := N_3(x, y) := \{u \in V(H) : \exists w \in N_2 \text{ s.t. } \deg_{B_w}(u) \geq 20\}
\]
and let \( G_3 := G_3(x, y) \) be the graph whose vertex set is \( N_2 \cup N_3 \) and whose edge set is given by

\[ E(G_3) := \{uw : u \in N_3, w \in N_2, \text{ and } \deg_{B_w}(u) \geq 20\} \]

Fix \( w \in N_2 \). Then

\[ \deg_{G_2}(w) \cdot \frac{\delta^*_2(H) \geq \beta n}{\beta n} \leq e(B_w) \leq 20 \cdot s + (n - s) \deg_{G_2}(w) \]

where \( s \) denotes the number of vertices \( u \in V(H) \) satisfying \( \deg_{B_u}(u) < 20 \). Then, for \( n \) sufficiently large

\[ n - s \geq \frac{20 \cdot s}{\deg_{G_2}(w)} \geq \beta n - \frac{20 \cdot n}{\log n} \geq \beta n/2 \]

As \( \deg_{G_3}(w) = n - s \) it follows that

\[ \deg_{G_3}(w) \geq \beta n/2 \text{ for every } w \in N_2; \] (3.6)

in particular this implies

\[ |N_3| \geq \beta n/2 \text{ and } e(G_3) \geq \frac{1}{2} \sum_{w \in N_2} \deg_{G_3}(w) \geq |N_2| \beta n/4 \geq \beta^2 n^2/8 \] (3.7)

This concludes the definition of \( \{x, y\}\)-cascade and required properties thereof.

**LINKS.** In addition to \( \{x, y\} \) and \( C(x, y) \) defined above let \( \{x', y'\} \) be a pair of vertices disjoint from \( \{x, y\} \), and let \( C(x', y') \) be an \( \{x', y'\}\)-cascade in \( H \). A quadruple \( (z, u, v, w) \in N_2(x, y) \times N_3(x, y) \times N_3(x', y') \times N_2(x', y') \) is said to be an \( \{(x, y), \{x', y'\}\}\)-link (in \( H \)) if

(L.1) \( x, y, z, u, v, w, y', x' \) are all distinct.

(L.2) \( \{z, u, v\}, \{u, v, w\} \in E(H) \), and

(L.3) \( zu \in E(G_3(x, y)) \) and \( vw \in E(G_3(x', y')) \).

**LEMMA 3.8.** If two distinct pairs of vertices namely \( \{x, y\} \) and \( \{x', y'\} \) admit an \( \{(x, y), \{x', y'\}\}\)-link, then \( H \) admits a 10-path connecting \( \{x, y\} \) and \( \{x', y'\} \).

**Proof.** Let \( (z, u, v, w) \in N_2(x, y) \times N_3(x, y) \times N_3(x', y') \times N_2(x', y') \) be an \( \{(x, y), \{x', y'\}\}\)-link. First we construct a 5-path connecting \( \{x, y\} \) and \( \{z, u\} \) through \( C(x, y) \). Having \( zu \in E(G_3(x, y)) \) means that there are at least 20 vertices \( z' \in N_{G_2}(x, y)(z) \subseteq N_1(x, y) \) such that \( \{z', z, u\} \in E(H) \). Hence we can choose one such \( z' \notin \{x, y, x', y', z, u, v, w\} \). The vertex \( z' \) lies in \( N_1(x, y) \); having \( zz' \in E(G_2) \) implies that \( \{y, z', z\} \in E(H) \). The 5-path is made complete with the fact that \( \{x, y, z'\} \in E(H) \). Let \( P \) denote this path.

It remains to construct a 5-path through the cascade of \( \{x', y'\} \) connecting \( \{v, w\} \) and \( \{x', y'\} \) and that this path meets no vertex of \( P \). The same argument used for constructing \( P \) can be used here as well albeit one change. We require a vertex \( z'' \in N_2(x', y') \) to play the corresponding rôle played by \( z' \) above and \( z'' \) must satisfy \( z'' \notin \{x, y, z', x', y', z, u, v, w\} \) (i.e., one has to avoid \( z' \) as well). Clearly there is enough freedom to do so. \[ \blacksquare \]
We are now ready to prove Lemma 1.5.

**Proof of Lemma 1.5.** Given \( d := d_{1.5}, \beta := \beta_{1.5} \), set

\[
\varrho_{1.5}(d, \beta) := \min \left\{ \frac{d^{\beta} 2^{15}}{12(1 - 1/(d - \beta))} \right\}.
\]

The second term here is incurred by the need to have \( H_\beta \) non-trivial according to (2.5) and retain \( e(H_\beta) = \Omega(n^3) \) as only \( o(n^3) \) edges are removed from \( H \) in order to obtain \( H_\beta \). Let \( \rho < \varrho_{1.5}(d, \beta) \), let \( H \) be \((\rho, d)\kappa\)-dense and let \( \{x, y\} \) and \( \{x', y'\} \) be two disjoint \( \beta \)-relevant pairs of \( H \).

By Lemma 3.8 it suffices to show that the cascades \( \mathcal{C}(x, y) \) and \( \mathcal{C}(x', y') \) taken in \( H_\beta \) admit an \((\{x, y\}, \{x', y'\})\)-link in \( H \). Owing to (3.7), \( e(G_3) \geq \beta^2 n^2/8 \); then there exists a subgraph \( F \subseteq G_3 \) satisfying \( \delta(F) \geq \beta^2 n/8 \) (see, e.g., [9, Proposition 1.2.2]). Then

\[
|V(F) \cap N_3| \geq \beta^2 n/8.
\]  

(3.10)

Indeed, all edges in \( G_3 \) (and thus in \( F \)) are of the form \( N_2 \times N_3 \) (though \( N_2 \cap N_3 \) need not be empty). Hence there is a vertex \( w \in N_2 \cap V(F) \); by definition \( N_F(w) \subseteq N_{G_3}(w) \subseteq N_3 \) and (3.10) follows.

Set

\[
B := \{(z, u) \in (V(F) \cap N_3) \times V(H) : z \neq u \text{ and } \deg_H(z, u, F) < d\beta^2 n/64\}.
\]

By Lemma 2.1 applied with \( G = F \), the vertex-cover of \( F \) namely \( Y := V(F) \cap N_3, k = \beta^2 n/8, \) and \( \Delta := d\beta^2 n/64 \) we arrive at

\[
|B| \leq \frac{gn^3}{d\beta^2 n/(2 \cdot 8) - 2 \cdot \Delta} = \frac{gn^3}{d\beta^2 n/16 - d\beta^2 n/32} = \frac{32g}{d\beta^2} \cdot n^2.
\]

A symmetrical argument applied to \( \mathcal{C}(x', y') \) asserts that the set

\[
B' := \{(z, u) \in (V(F') \cap N'_3) \times V(H) : z \neq u \text{ and } \deg_H(z, u, F') < d\beta^2 n/64\}
\]

satisfies \( |B'| \leq \frac{32g}{d\beta^2} \cdot n^2 \) as well, where here \( F' \subseteq G'_3 \) is the counterpart of \( F \) in this argument (i.e., it is a subgraph of \( G'_3 \) satisfying \( \delta(F') \geq \beta^2 n/8 \)).

By (3.10) the set \( (V(F) \cap N_3) \times (V(F') \cap N'_3) \) has size at least \( \beta^4 n^2/2^6 \); removing degenerate members (i.e., members of the form \( (x, x) \)) we retain at least \( \beta^4 n^2/2^7 \) non-degenerate members. The latter set of non-degenerate pairs gives rise to an unordered set of pairs of size at least \( \beta^4 n^2/2^8 \).

Let \( T \) denote the unordered set of pairs thus formed. Then

\[
|T \setminus (\tilde{B} \cup \tilde{B}')| \geq \beta^4 n^2/2^8 - \frac{64g}{d\beta^2} n^2 \geq \beta^4 n^2/2^9.
\]

where \( \tilde{B} \) and \( \tilde{B}' \) denote the underlying unordered sets arising from \( B \) and \( B' \), respectively. For \( n \) sufficiently large the set \( T \setminus (B \cup B') \) is non-empty. Each member \( \{u, v\} \in T \setminus (B \cup B') \) satisfies \( u \neq v, u \in V(F) \cap N_3, v \in V(F') \cap N'_3, \deg_G(u, v, F) \geq d\beta^2 n/64, \) and \( \deg_G(v, u, F') \geq d\beta^2 n/64 \).

That is there are at least \( d\beta^2 n/64 \) edges \( \{u, v, z\} \in E(H) \) with \( uz \in E(F) \) (so that \( z \in N_2 \)) and at least \( d\beta^2 n/64 \) edges \( \{u, v, w\} \in E(H) \) with \( vw \in E(F') \) (so that \( w \in N'_2 \)). Hence for \( n \) sufficiently large we may insist on (many choices) \( w \neq z \) and thus form the required \( \{(x, y), (x', y')\}\)-link. \( \blacksquare \)
§4. The absorbing-path lemma

Let $H$ be a 3-graph. For $\beta > 0$ and $v \in V(H)$ a quadruple $(x, y, z, w) \in V(H)^4$ is said to be a $(\beta, v)$-absorber if

(A.1) $\{x, y, z\}, \{y, z, w\}, \{v, x, y\}, \{v, y, z\}, \{v, z, w\} \in E(H)$. 

(A.2) $\{x, y, z\}, \{y, z, w\} \in E(H_\beta)$. 

We say $\beta$-absorber to mean $(\beta, v)$-absorber for some $v \in V(H)$. Write $L_v := L_v(H)$ to denote the link graph of $v$, that is the graph whose vertex set is $V(H) \setminus \{v\}$ and where two (distinct) vertices, namely $x$ and $y$, form an edge in $L_v$ provided that $(x, y, v) \in E(H)$. Put $L_{\beta,v} := L_v(H_\beta) \subseteq L_v$. 

To prove the absorbing path lemma, namely Lemma 1.6, we have a three step argument modelled after [34]: first we establish a counting result for $(\beta, v)$-absorbers per vertex $v$ (Lemma 4.1); second, we prove the existence of a "small" set $F$ of disjoint of $\beta$-absorbers that can service a "small" yet arbitrary number of vertices (Lemma 4.3); third, we "string" the members of $F$ into a single path yielding the required absorbing path. Throughout the proof of the absorbing-path lemma the following lemma is the sole part due to which a Dirac-type condition is imposed.

**Lemma 4.1.** For every $d_{4,1}, \alpha_{4,1}, \beta_{4,1} \in (0, 1]$ such that $\beta_{4,1} < d_{4,1}$ there exist an $n_{4,1}$, a $\varrho_{4,1} = \varrho_{4,1}(d_{4,1}, \alpha_{4,1}, \beta_{4,1}) > 0$, and a $c_{4,1} := c_{4,1}(d_{4,1}, \alpha_{4,1}) := \frac{d_{4,1}^2 n_{4,1}^6}{2^{10}}$ such that the following holds for any integer $n \geq n_{4,1}$ and $\varrho < \varrho_{4,1}$. Let $H$ be an $n$-vertex $(\varrho, d_{4,1})$-dense 3-graph satisfying $\delta(H) \geq \alpha_{4,1} n_{4,1}^3$, and let $v \in V(H)$. Then there are at least $c_{4,1} n_{4,1}^3 (\beta, v)$-absorbers in $H$.

**Proof.** Given $\alpha := \alpha_{4,1}, \beta := \beta_{4,1}$ and $d := d_{4,1}$ set

$$\varrho_{4,1} := \min\left\{\frac{\alpha d^2}{2^{10}}, \alpha (d - \beta)/8\right\}$$

(4.2) let $\varrho < \varrho_{4,1}$ and let $n$ be sufficiently large. Let $H$ be an $n$-vertex $(\varrho, d)$-dense 3-graph as prescribed and fix $v \in V(H)$.

Having $\deg_H(v) \geq \alpha(n_{4,1}^3)$ asserts that $e(L_v) \geq \alpha(n_{4,1}^3)$. Then for $n$ sufficiently large

$$e(L_{\beta,v}) \geq e(L_v) - |B_\beta| \geq \alpha \left(\frac{n_{4,1}^3}{2}\right) - \frac{\varrho}{d - \beta} n_{4,1}^2 \geq \alpha n_{4,1}^2/8,$$

where $B_\beta$ is as in (2.3). Sidorenko’s conjecture [11, 38] is true for the 2-graph $P_4$ [2] which is the path consisting of 3 edges and 4 vertices. Then for $n$ sufficiently large there are at least $(n - 1)^4 \left(\frac{2e(L_{\beta,v})}{n^2}\right)^3 \geq \frac{\alpha^3 n^4}{2^{10}}$ homomorphisms of $P_4$ into $L_{\beta,v}$. Consequently (and again assuming $n$ is sufficiently large) there is a collection $\mathcal{P}$ of at least $\alpha^3 n^4/2^{10}$ labelled copies of $P_4$ in $L_{\beta,v}$. For an ordered pair $(u, w) \in V(L_{\beta,v}) \times V(L_{\beta,v})$ let $P_4(u, w)$ denote the number of members of $\mathcal{P}$ of the form $(x, u, w, y)$.

Set $\Delta := \alpha^3 n^2/2^{12}$. Let $X := \{(u, w) \in V(L_{\beta,v}) \times V(L_{\beta,v}) : P_4(u, w) < \Delta\}$ and let $Y := V(L_{\beta,v}) \setminus X$. In preparation for two applications of Lemma 2.1 we define three graphs, namely $G_1$, $G_2$, and $G_3$, as follows. Let $\tilde{Y}$ denote the set of unordered pairs underlying $Y$ and set $G_2 := (V(L_{\beta,v}), \tilde{Y})$. For $(u, w) \in Y$, set $A_{(u,w)} := \{(a, u) : (a, u, w, b) \in \mathcal{P}\}$ and set $B_{(u,w)} := \{(w, b) : (a, u, w, b) \in \mathcal{P}\}$. Define $G_1 := (V(L_{\beta,v}) \cup_{(u,w) \in Y} \tilde{A}_{(u,w)}$ where $\tilde{A}_{(u,w)}$ is the set of unordered pairs underlying $A_{(u,w)}$. In a similar manner, define $G_3 := (V(L_{\beta,v}) \cup_{(u,w) \in Y} \tilde{B}_{(u,w)})$
where $\tilde{B}_{(u,w)}$ is the set of unordered pairs underlying $B_{(u,w)}$. The graphs $G_1, G_2, G_3$ are not necessarily edge disjoint. In addition define the sets

$$U := \{u \in V(L_v) : (a, u, w, b) \in \mathcal{P} \text{ for some } a, w, b \in V(L_{\beta,v})\}$$

$$W := \{w \in V(L_v) : (a, u, w, b) \in \mathcal{P} \text{ for some } a, u, b \in V(L_{\beta,v})\};$$

observe that $U$ is a vertex cover of $G_1$ and that $W$ is a vertex cover of $G_3$.

Consider

$$\frac{\alpha^3}{2^{10}} n^4 \leq |\mathcal{P}| = \sum_{(u,w) \in V(L_{\beta,v})^2} P_4(u,w) \leq |X|\Delta + (n^2 - |X|)n^2;$$

and isolate $|X|$ thereof; one arrives at $|X| < \frac{1-\alpha^3}{1-\alpha^3/2^{10}} n^2$. As $\frac{1-\alpha^3}{1-\alpha^3/2^{10}} \leq 1 - \frac{\alpha^3}{2^{12}}$ it follows that $|Y| > \frac{\alpha^3 n^2}{2^{12}}$ so that $e(G_2) \geq \frac{\alpha^3 n^2}{2^{13}}$. For $(u, w) \in Y$ observe that $|A_{(u,w)}| > \frac{\alpha^3 n^2}{2^{13}}$; for if one of these sets, say $A_{(u,w)}$, violates this inequality then $P_4(u,w) \leq |A_{(u,w)}|\deg_{L_{\beta,v}}(w) < \frac{\alpha^3}{2^{12}} n \cdot n < \Delta$. Consequently, $\deg_{G_1}(u), \deg_{G_3}(w) \geq \frac{\alpha^3 n^2}{2^{13}}$ for every $u \in U$ and every $w \in W$, respectively.

Set

$$B_U := \{(u,w) \in U \times W : \deg_H(u,w,G_1) < \alpha^3 n/2^{16}\},$$

$$B_W := \{(u,w) \in U \times W : \deg_H(u,w,G_3) < \alpha^3 n^2/2^{14}\}. $$

Then by Lemma 2.1 for $G_1, U$, and $B_U$ and for $G_3, W$, and $B_W$ we attain $|B_U|, |B_W| \leq \frac{215}{\alpha \delta^*} n^2$.

This implies that in $G_2$ there are at least $e(G_2) - |B_U| - |B_W| \geq \alpha^3 n^2/2^{13} - \frac{216}{\alpha \delta^*} n^2 \geq \frac{\alpha^3 n^2}{2^{14}}$ unordered pairs $\{u, w\} \in E(G_2) \subseteq E(L_{\beta,v})$ with $u \in U$ and $w \in W$ such that $\deg_H(u,w,G_1), \deg_H(w,u,G_3) > \alpha^3 n/2^{16}$. Call these pairs in $E(G_2)$ good. Given one such good pair $(u,v) \in U \times W$ each of the $\deg_H(u,w,G_1)$ neighbours $a \neq w$ of $u$ in $G_1$ forms a triple $(a, u, w)$ which extends into at least $\deg_H(w,u,G_3) - 2 = \beta(v)-$absorbers $(a, u, w, b)$ with all members of the quadruple being distinct. Hence, for $n$ sufficiently large there are

$$(\alpha^3 n/2^{16} - 1) \cdot (\alpha^3 n/2^{16} - 2) \cdot \alpha^3 n^2/2^{14} \geq (\alpha^3 n/2^{17})^2 \alpha^3 n^2/2^{14} \beta(v)-\text{absorbers concluding the proof.}$$

Let $H$ be a 3-graph. For $v \in V(H)$ and $\beta > 0$ let $A_{\beta,v}$ denote the set of $(\beta,v)$-absorbers in $H$.

**Lemma 4.3.** For every $d_{4,3}, \alpha_{4,3}, \beta_{4,3} \in (0,1]$ such that $\beta_{4,3} < d_{4,3}$ there exist an integer $n_{4,3}$, and reals $\varrho_{4,3} := d_{4,3}(\alpha_{4,3}, \alpha_{4,3}, \beta_{4,3}) > 0$, $0 < f_{4,3} := f_{4,3}(d_{4,3}, \alpha_{4,3}, \beta_{4,3}) \leq \frac{d_{4,3}^2}{17\delta^*}$, $a_{4,3} := a_{4,3}(d_{4,3}, \alpha_{4,3}, \beta_{4,3}) \in (0,1]$ such that the following holds whenever $n > n_{4,3}$ and $\varrho < \varrho_{4,3}$. Let $H$ be an $n$-vertex $(\varrho, d_{4,3})$-dense 3-graph satisfying $\delta(H) \geq \alpha_{4,3}\left(\frac{n}{2}\right)$. Then there exists a set $\mathcal{F}$ of vertex-disjoint $\beta$-absorbers such that

(F.1) $|\mathcal{F}| \leq f_{4,3}n$

(F.2) For every $v \in V(H)$: $|A_{\beta_{4,3},v} \cap \mathcal{F}| \geq a_{4,3}n$. 


Proof. Given \( d := d_{1.3}, \alpha := \alpha_{1.3}, \) and \( \beta := \beta_{1.3} \) set \( \varrho_{1.3} = \varrho_{4.1}(d, \alpha, \beta), \) set \( c := c_{1.1}(d, \alpha), \) and let \( \varrho < \varrho_{1.3}. \) Then \( |A_{\beta, v}| \geq cn^4 \) for every \( v \in V(H), \) by Lemma 4.1. Set

\[
\gamma := \frac{c}{4 \cdot 17}
\]

and let \( F' \) be a set of quadruples where each quadruple in \( V(H)^4 \) is put in \( F' \) independently at random with probability \( \gamma n^{-3}. \) Then \( E|F'| = \gamma n; \) Chernoff’s inequality \([17, Equation (2.9)]\) then yields that

\[
|F'| \leq 2\gamma n
\]

holds with high probability. Furthermore, \( E|A_{\beta, v} \cap F'| \geq cn^4 \gamma n^{-3} = c\gamma n \) for every vertex \( v. \) Chernoff’s inequality \([17, Equation (2.9)]\) and the union bound yield \( P\{\exists v \in V(H) : |A_{\beta, v} \cap F'| < c\gamma n/2\} = o(1). \) Hence, w.h.p.

\[
|A_{\beta, v} \cap F'| \geq c\gamma n/2, \text{ for every } v \in V(H).
\]

Let \( I := I(F') \) denote the number of pairs of members of \( F' \) that meet one another. Note that \( E[I] \leq n^4 \cdot 4 \cdot 4 \cdot n^3 \cdot (\gamma n^{-3})^2 \leq 16\gamma^2 n; \) hence, by Markov inequality we attain that

\[
|I| < 17\gamma^2 n
\]

holds with positive probability.

It follows then that an \( F' \) satisfying \( (4.5), (4.6), \) and \( (4.7) \) exists. Fix one such \( F'. \) Define \( F \) to be the set of quadruples attained from \( F' \) by, first, removing all quadruples which do not \( \beta \)-absorb any \( v \) and, second, from each intersecting pair of quadruples remove one of the members of that pair. Trivially, Property \( (F.1) \) holds for \( F. \) To see that Property \( (F.2) \) holds note that for every \( v \in V(H) \) we have

\[
|A_{\beta, v} \cap F| \geq c\gamma n/2 - 17\gamma^2 n \overset{(4.4)}{\geq} c\gamma n/4
\]

as required. \( \blacksquare \)

We are now ready to prove Lemma 1.6. All that remains is to "string" the members of \( F \) (from Lemma 4.3) into a single path and prove its absorption capabilities.

**Proof of Lemma 1.6.** Let \( d := d_{1.6}, \alpha := \alpha_{1.6}, d^2\alpha^9 \leq \beta := \beta_{1.6} < d \) be given. Set \( \kappa := \kappa_{1.6} := 10f_{1.3}(d, \alpha) \) (so that \( \kappa \ll \beta), \) \( m := m_{1.6} := m_{4.3}(d, \alpha), \) and

\[
\varrho_{1.6} := \min \left\{ \varrho_{1.5}(d, \beta - \kappa) \cdot (1 - \kappa)^3, \varrho_{4.3}(d, \alpha, \beta) \right\} .
\]

(4.8)

Let \( \varrho < \varrho_{1.6}, \) let \( n \) be sufficiently large and let \( H \) be an \( n \)-vertex \((\varrho, d)_\kappa\)-dense 3-graph with \( \delta(H) \geq \alpha(n^{-1}). \)

Let \( F \) denote the set of \( \beta \)-absorbers in \((F.1)\) existence of which in \( H \) is assured by Lemma 4.3 as \( \varrho < \varrho_{4.3}(d, \alpha, \beta). \) For \( F := (x, y, z, w) \in F \) we refer to \( \{x, y\} \) as the front end-pair of \( F \) and to \( \{z, w\} \) as the rear end-pair of \( F. \) Fix an arbitrary ordering on the members of \( F, \) namely, \( F_1, F_2, \ldots, F_r \) where \( r = |F| \leq f_{4.3}(d, \alpha)n. \) Below we prove that a path \( A \) of the form \( F_1 \circ C_1 \circ \cdots \circ F_{r-1} \circ C_{r-1} \circ F_r \) exists, where here each \( C_i \) is a 10-path connecting the rear end-pair of \( F_i \) with the front end-pair of \( F_{i+1}; \) we use \( \circ \) to denote path concatenations along pairs. For such an \( A \) observe, first, that \( |V(A)| = 4r + 6(r - 1) \leq 10r \leq \kappa n. \) Observe, second, that owing to \((F.2)\)
together with (what is by now) a standard greedy argument (see, e.g., [34, Claim 2.6]) it follows that such an $A$ can absorb any set of vertices of size at most $mn$.

To complete the proof of this lemma it remains to establish the existence of $A$. This we do inductively as follows. Put $A_1 := F_1$ and suppose that the (partial) path $A_i := F_1 \circ C_1 \circ \cdots \circ F_{i-1} \circ C_{i-1} \circ F_i$ has been defined for some $i \in [r-1]$. Set $V_i := (V(H) \setminus (V(A_i) \cup V(F))) \cup \{a, b, c, d\}$ where \{a, b\} is the rear end-pair of $F_i$ and \{c, d\} is the front end-pair of $F_{i+1}$. As $|V(A_i) \cup V(F)| \leq \kappa n$ it follows that $|V_i| \geq (1 - \kappa) n$ for every $i \in [r]$. Owing to (4.8) and the hereditary nature of $\kappa$-denseness over induced subgraphs of $H$ asserts that $H[V_i]$ is $(\rho_{1.5}(d, \beta - \kappa), d)$-dense. Indeed, fix $G_1, G_2 \subseteq V_i \times V_i$. Then

$$e_{H[V_i]}(G_1, G_2) \geq d \mathcal{P}_2(G_1, G_2) - \rho n^3 \geq d \mathcal{P}_2(G_1, G_2) - \rho_{1.5}(d, \beta - \kappa)(1 - \kappa)^3 n^3 \geq d \mathcal{P}_2(G_1, G_2) - \rho_{1.5}(d, \beta - \kappa)|V_i|^3.$$  

Consequently, any two $(\beta - \kappa)$-relevant pairs of $H[V_i]$ can be connected via a 10-path in $H[V_i]$ by Lemma 1.5. This in particular holds for \{a, b\} and \{c, d\} which are both $\beta$-relevant in $H$ and thus both are $(\beta - \kappa)$-relevant in $H[V_i]$, by Observation 2.7. This implies that $C_{i+1}$ as defined above exists and consequently $A_{i+1}$ exists as well.  

\section{Path-cover lemma}

In this section we prove our path-cover lemma, i.e., Lemma 1.8. We require some preparation. A 3-graph $H$ is said to be $t$-partite if there is a vertex partition $V(H) = V_1 \cup V_2 \cup \cdots \cup V_t$ such that each $e \in E(H)$ satisfies $|e \cap V_i| \leq 1$ whenever $i \in [t]$. We say that $H$ is $t$-partite equitable if in addition $|V_1| \leq |V_2| \leq \cdots \leq |V_t| \leq |V_1| + 1$ holds. We also refer to the partition itself as equitable. An $n$-vertex 3-partite 3-graph $H$ with an underlying partition $V(H) = X \cup Y \cup Z$ is said to be $\varepsilon$-regular if

$$e_H(X', Y', Z') = \frac{e_H(X, Y, Z)}{|X||Y||Z|} |X'||Y'||Z'| \pm \varepsilon n^3$$  

holds for every $X' \subseteq X$, $Y' \subseteq Y$, and $Z' \subseteq Z$. If only the lower bound implied in (5.1) is assumed we say such an $H$ is $\varepsilon$-lower-regular. If in addition $e_H(X, Y, Z)/|X||Y||Z| \geq d$ then we say $H$ is $(\varepsilon, d)$-regular or $(\varepsilon, d)$-lower-regular, respectively. The following result is a commonly known generalisation of the main result of [39].

\begin{lemma} \textbf{(Weak-regularity lemma for 3-graphs [39])} \end{lemma}

For every $\varepsilon_{5.2} > 0$ and integer $t_{5.2}$ there exist integers $n_{5.2}$ and $T_{5.2}$ such that the following holds whenever $n \geq n_{5.2}$. Let $H$ be an $n$-vertex 3-graph. Then there exists an integer $t$ satisfying $t_{5.2} \leq t \leq T_{5.2}$ and an equitable partition $V(H) = V_1 \cup V_2 \cup \cdots \cup V_t$ such that for all but at most $\varepsilon^3$ triples $i, j, k \in [t]$ the sets $V_i, V_j, V_k$ induce an $\varepsilon_{5.2}$-regular 3-partite 3-graph denoted $H[V_i, V_j, V_k]$.

Given a 3-graph $H$ regularised per Lemma 5.2 and a real $d > 0$ define $R_d := R_d(H)$ to denote the 3-graph whose vertices are the clusters (i.e., sets) $(V_i)_{i \in [t]}$ and whose edges are the triples $(V_i, V_j, V_k)$, $i, j, k \in [t]$, such that $H[V_i, V_j, V_k]$ is $(\varepsilon, d)$-regular. It will be convinient to identify $V(R_d)$ with $[t] := \{1, \ldots, t\}$. Given $X \subseteq V(R_d)$ define $\cup X := \bigcup_{i \in X} V_i$. An edge $e \in E(H)$ is said to be crossing with respect to $X$ if there are three clusters $V_i, V_j, V_k$ captured by $X$ such that $|e \cap V_i| = 1$, $|e \cap V_j| = 1$, and $|e \cap V_k| = 1$.

\begin{lemma} \textbf{(Path packing lemma [35, Claim 4.2])} \end{lemma}

For all $0 < \varepsilon < d < 1$, every $(\varepsilon, d)$-lower-regular 3-partite equitable 3-graph $H$ on $n$ vertices, $n$ sufficiently large, contains a family $P$ of vertex
disjoint-paths such that for each \( P \in \mathcal{P} \) we have \( |V(P)| \geq \varepsilon(d-\varepsilon)n/3 \) and \( \sum_{P \in \mathcal{P}} |V(P)| \geq (1-2\varepsilon)n \).

The following is a triviality whose proof is included for completeness.

**Lemma 5.4.** For all \( d_{5.4} > 0 \) and \( \zeta_{5.4} > 0 \) and

\[
\rho < \rho_{5.4}(d_{5.4}, \zeta_{5.4}) := \frac{d_{5.4} \cdot (\zeta_{5.4})^3}{27} \tag{5.5}
\]

the following holds. Let \( H \) be a \((\rho, d_{5.4})\)-dense 3-graph. Then \( H \) admits a matching covering all but at most \( \max\{2, \zeta_{5.4}|V(H)|\} \) vertices.

**Proof.** Write \( d := d_{5.4} \) and \( \zeta := \zeta_{5.4} \). Let \( M \) be a maximum matching in \( H \). Let \( X := V(H) \setminus V(M) \) denote the set of vertices not covered by the members of \( M \). If \( |X| = 2 \), then we are done. Assume then that \( |X| \geq 3 \) in which case \( e_H(X) = 0 \) by the maximality of \( M \). Then

\[
d \left( \frac{|X|}{3} \right) - \rho |V(H)|^3 \leq e_H(X) \leq 0.
\]

Consequently

\[
d \frac{|X|^3}{27} \leq \rho |V(H)|^3.
\]

Assuming that \( |X| > \zeta |V(H)| \) we arrive at

\[
\zeta |V(H)| < |X| \leq (27 \cdot \rho d^{-1})^{1/3} |V(H)|
\]

contradicting (5.5). Consequently, in this case, \( |X| \leq \zeta |V(H)| \) must hold. \hfill \Box

We are now ready to prove our path-cover lemma, namely Lemma 1.8.

**Proof of Lemma 1.8.** Given \( d := d_{1.8} \) and \( \zeta := \zeta_{1.8} \) let \( \rho' := \rho_{5.4}(d/2, \zeta/12) \) and set

\[
t_{\text{reg}} := \max\{8/\rho', 8/\zeta\}, \quad d' := \rho'/4, \quad \varepsilon_{\text{reg}} := \min\{d'/2, \zeta/24\}. \tag{5.6}
\]

In addition, set

\[
\rho_{1.8} := \rho'/4 \quad \text{and} \quad \gamma_{1.8} := \frac{T_{5.2}(\varepsilon_{\text{reg}}, t_{\text{reg}})}{\varepsilon_{\text{reg}}(d' - \varepsilon_{\text{reg}})}. \tag{5.7}
\]

Let \( n \) be sufficiently, let \( \rho < \rho_{1.8} \), and let \( H \) be an \( n \)-vertex \((\rho, d')\)-dense 3-graph.

Let \( R_{d'} := R_{d'}(H) \) denote the reduced graph of \( H \) obtained after regularising \( H \) using the weak-regularity lemma, namely Lemma 5.2, applied with \( \varepsilon_{\text{reg}} \) and \( t_{\text{reg}} \). Let \( |V(R_{d'})| = t \) and identify \( V(R_{d'}) \) with \( \{t\} \).

\( R_{d'} \) is \((\rho', d/2)\)-dense. \tag{5.8}

To see (5.8) fix \( X \subseteq V(R_{d'}) \) and let \( C_X \) denote the number of edges of \( H \) which are crossing with respect to \( X \) and that lie in \((\varepsilon_{\text{reg}}, d')\)-regular triple \( H[V_i, V_j, V_k] \) where the sets \( V_i, V_j, V_k \) are taken from the underlying regularity partition. Then \( e_{R_{d'}}(X) \geq C_X/2(n/t)^3 \); the factor 2 appearing here is to cope with the the fact that cluster sizes are in the set \( \{n/t, n/t + 1\} \); we use the fact that for \( n \) sufficiently large \( 2(n/t)^3 \geq (n/t + 1)^3 \).

\[
C_X \geq e_H(\cup X) - |X| \cdot 2(n/t)^3 - |X|^2 \cdot 2(n/t)^3 - \varepsilon_{\text{reg}}t^3 \cdot 2(n/t)^3 - |X|^3 d' \cdot 2(n/t)^3;
\]

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this lower bound on \(C_X\) is attained by, first, removing from \(E(H[\cup X])\) all edges that have at least two of their vertices in the same cluster captured by \(X\); second, removing all (crossing) edges found in \(\varepsilon_{\text{reg}}\)-irregular triples of clusters; thirdly, removing all (crossing) edges found in triples of clusters whose edge density is at most \(d'\). As \(|X| \leq t\) we arrive at

\[
e_{R_{d'}}(X) \geq \frac{e_H(\cup X)}{2(n/t)^3} - t - t^2 - \varepsilon_{\text{reg}}t^3 - d't^3.
\]

As \(H\) is \((\varrho, d)\)-dense

\[
e_H(\cup X) \geq d\left(\sum_{i \in X} \frac{|V_i|}{3}\right) - \varrho n^3 \geq d\left(\frac{|X|}{3}\right)(n/t)^3 - \varrho n^3;
\]

here the term \(\left(\frac{|X|}{3}\right)(n/t)^3\) accounts only for edges crossing with respect to \(X\) while \(\left(\sum_{i \in X} |V_i|\right)\) accounts also for triples inside clusters captured by \(X\); hence the inequality. By (5.6), \(\varepsilon_{\text{reg}} \leq d' = \varrho'/4\) and \(d' \leq \varrho'/4\). Owing to \(t \geq t_{\text{reg}} \geq 8/\varrho'\), again by (5.6), then \(t + t^2 \leq 2t^2 \leq \varrho't^3/4\). In addition, by (5.7), \(\varrho \leq \varrho'/4\). Then

\[
e_{R_{d'}}(X) \geq \frac{d}{2}\left(\frac{|X|}{3}\right) - (\varrho + \varrho'/4 + \varepsilon_{\text{reg}} + d')t^3 \geq \frac{d}{2}\left(\frac{|X|}{3}\right) - \varrho't^3;
\]

concluding the proof of (5.8).

In view of (5.8) and the choice of \(\varrho'\) it follows, by Lemma 5.4, that \(R_{d'}\) admits a matching \(M\) missing at most \(\max\{2, \zeta t/12\}\) vertices of \(R_d\). For each edge \((V_i, V_j, V_k)\) of \(M\) apply Lemma 5.3 to \(H[V_i, V_j, V_k]\) to as to obtain a system of vertex-disjoint paths as described in Lemma 5.3. Let \(\mathcal{P}\) denote the system of paths thus generated in \(H\) over all edges of \(M\). In each \(H[V_i, V_j, V_k]\) corresponding to an edge \((V_i, V_j, V_k)\) of \(M\) at most \(\frac{|X|}{\varepsilon_{\text{reg}}(d' - \varepsilon_{\text{reg}})}\) paths are packed. As \(|M| \leq T_{5.2}(\varepsilon_{\text{reg}}, t_{\text{reg}})/3\) at most \(3\gamma_{1.8}\) paths are thus packed.

It remains to argue that the members of \(\mathcal{P}\) cover all but at most \(\zeta n\) vertices of \(H\). In each \(H[V_i, V_j, V_k]\) corresponding to an edge \((V_i, V_j, V_k)\) of \(M\) at most \(2\varepsilon_{\text{reg}} \cdot 6n/t\) vertices of \(H[V_i, V_j, V_k]\) are missed. As \(|M| \leq t/3\), at most \(12\varepsilon_{\text{reg}} n\) vertices of \(H\) are missed this way. From the clusters not covered by \(M\) at most \(\max\{2, \zeta t/12\}\cdot 2n/t\) vertices of \(H\) are missed. Overall at most \(12\varepsilon_{\text{reg}} + \max\{4/t, \zeta/2\}\) \(n\) vertices of \(H\) are missed. Owing to (5.6), \(12\varepsilon_{\text{reg}} \leq \zeta/2\) and \(t \geq t_{\text{reg}} \geq 8/\zeta\) (so that \(4/t \leq \zeta/2\)); consequently \(12\varepsilon_{\text{reg}} + \max\{12/t, \zeta/2\} \leq \zeta\) as required.

\section*{§6. Proof of Theorem 1.4}

We shall require the following variant of [34, Lemma 2.7]; commonly referred to as the reservoir lemma. For us this lemma is a straightforward application of Chernoff’s inequality. Proof included for completeness.

\textbf{Lemma 6.1.} For every triple of reals \(\nu, \kappa, \beta \in (0, 1]\) satisfying \(\kappa \leq \beta/4\) and \(\nu \leq 1/2\) the following holds for all sufficiently large integers \(n\). Let \(H\) be an \(n\)-vertex 3-graph with \(H_\beta\) non-trivial \(i.e., \delta_2(H_\beta) \geq \beta n\) and let \(A \subseteq V(H)\) have \(|A| = \kappa n < n/4\). Then there exists a set \(R \subseteq V(H) \setminus A\) of size \(|R| = \lceil \nu n \rceil\) satisfying \(|N_{H_\beta}(x, y) \cap R| \geq \beta 

\text{Proof.} Let \(\nu, \kappa, \beta, n, H, A\) be given and let \(n\) be sufficiently large. As \(|\nu n| \leq n/2 + 1\) and \(|A| \leq n/4\) the set \(\binom{V(H)}{\lfloor \nu n \rfloor}\) is non-empty for \(n\) sufficiently large. Let \(R \subseteq \binom{V(H)}{\lfloor \nu n \rfloor}\) be chosen uniformly at
random. Fix a $\beta$-relevant pair $\{x, y\} \in \binom{V(H)}{2}$. Then the random variable $X(x, y) := |N_{H_\beta}(x, y) \cap R|$ has the hypergeometric distribution with

$$\mathbb{E}X(x, y) \geq \frac{(|N_{H_\beta}(x, y)| - |A|)[\nu n]}{n - |A|} \geq \frac{(\beta - \kappa)\nu n^2}{n} \geq (\beta - \kappa)\nu n \geq \beta\nu n/2.$$ 

By Chernoff’s inequality [17, Equation (2.9)], $\mathbb{P}[X(x, y) \leq 2^{-1}\beta\nu n/2] \leq 2^{-\Omega(n)}$. As there are at most $n^2\beta$-relevant pairs in $H$ the claim follows whenever $n$ is sufficiently large. 

We are now ready to prove our main result, namely Theorem 1.4.

**Proof of Theorem 1.4.** Given $d$ and $\alpha$ set

$$\kappa := \kappa_1(d, \alpha), \ m := m_1(d, \alpha), \ \nu := m/2.$$ 

(6.2)

Select $d^2\alpha^9 < \beta < d$ and recall from Lemma 1.6 that we may insist on $\kappa \ll \beta$. Set

$$\varrho := \min \left\{ q_{1.6}(d, \alpha, \beta), 2^{-1}q_{1.8}(d, m/2)(d - \beta)(1 - \kappa - \nu)^3, q_{1.5}(d, \beta\nu/8)(\nu/2)^3 \right\}.$$ 

(6.3)

Let $n$ be sufficiently large and let $H$ be an $n$-vertex $(\varrho, d)\kappa$-dense 3-graph satisfying $\delta_1(H) \geq \alpha(n - 1)^2$. As $\varrho \leq q_{1.6}(d, \alpha, \beta)$, then owing to Lemma 1.6, $H$ admits an $m\nu$-absorbing $\kappa\nu$-path $A$. By Lemma 6.1 there exists a set $R \subseteq V(H) \setminus V(A)$ with $|R| = \lfloor \nu n \rfloor$ satisfying

$$|N_{H_\beta}(x, y) \cap R| \geq \beta\nu n/4 \text{ for all } \beta\text{-relevant pairs } x, y \in V(H).$$ 

(6.4)

Define $H' := H - V(A) - R$. Then $|V(H')| \geq (1 - \kappa - \nu)n$ and is $(2^{-1}q_{1.8}(d, m/2)(d - \beta), d)\kappa$-dense. To see the latter fix $G_1, G_2 \subseteq V(H') \times V(H')$ and note that as $H'$ is an induced subgraph of $H$ then

$$e_{H'}(G_1, G_2) \geq d|P_2(G_1, G_2)| - \varrho n^3 \geq d|P_2(G_1, G_2)| - 2^{-1}q_{1.8}(d, m/2)(d - \beta)(1 - \kappa - \nu)^3n^3 \geq d|P_2(G_1, G_2)| - 2^{-1}q_{1.8}(d, m/2)(d - \beta)|V(H')|^3;$$

as required. Consequently $H'_\beta$ is $(q_{1.8}(d, m/2), d)$-dense. Indeed, fix $X \subseteq V(H'_\beta) = V(H')$ and note that

$$e_{H'_\beta}(X) \geq e_{H'}(X) - |B_{H'}(H')| \cdot |V(H')|$$

$$= e_{H'}(X) - |B_{H'_\beta}(H')| \cdot |V(H')|$$

(2.4)

$$\geq d\left(\frac{|X|}{3}\right) - \frac{q_{1.8}(d, m/2)(d - \beta)}{2}|V(H')|^3 - \frac{q_{1.8}(d, m/2)(d - \beta)}{2(d - \beta)}|V(H'_\beta)|^3$$

(6.4)

$$\geq \frac{d - \beta \leq 1}{d\left(\frac{|X|}{3}\right) - \frac{q_{1.8}(d, m/2)}{2}|V(H')|^3}$$

as required. By Lemma 1.8, $H'_\beta$ admits a collection $\mathcal{P}' = \{P_1, \ldots, P_{\gamma - 1}\}$, $\gamma - 1 \leq q_{1.8}(d, m/2)$, of vertex-disjoint paths covering all but at most $m|V(H')|/2 \leq m\nu n$ vertices of $H'$. Write $P_{\gamma} := A$ and set $\mathcal{P} := \mathcal{P}' \cup \{P_{\gamma}\}$. 

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Using $R$ and $\gamma$ applications of the connecting lemma, namely Lemma 1.5, we concatenate the members of $\mathcal{P}$ into a (tight) cycle. We do so in two steps. First we construct the path $L := P_1 \circ C_2 \circ P_2 \circ C_3 \circ \cdots \circ C_j \circ P_\gamma$ where each $C_i$ is a 10-path and $\bigcup V(C_i) \setminus V(\mathcal{P}) \subseteq R$. Second we connect the remaining free end-pair of $P_\gamma$ with the remaining free end-pair of $P_1$ with an additional path $C_j$ satisfying $V(C_j) \setminus V(\mathcal{P}) \subseteq R$ as well; the resulting cycle we denote by $L_\gamma$.

The construction of $L_\gamma$ is done inductively as follows. Set $L_1 := P_1$. Assuming $L_i := P_1 \circ C_\gamma \cdots \circ C_i \circ P_1$ has been defined for $i \in \lbrack \gamma - 1 \rbrack$ we define $L_{i+1}$ as follows. Let $\{a, b\}$ be the free end-pair of $P_i$ and let $\{c, d\}$ be one of the end pairs of $P_{i+1}$. Set $R_i := (R \setminus V(L_i)) \cup \{a, b, c, d\}$. Then for $n$ sufficiently large $|R_i| \geq |R| - 10\gamma \geq \nu n/2$. By (6.3), $\rho \leq \rho_{1.5}(d, \beta\nu/8)(\nu/2)^3$; consequently $H[R_i]$ is $(\rho_{1.5}(d, \beta\nu/8), d)_\mathcal{A}$-dense. We seek to apply the connecting lemma to $\{a, b\}$ and $\{c, d\}$ in $H[R_i]$ and to that end seek to prove that both these pairs are $\beta\nu/8$-relevant in $H[R_i]$; i.e., that these two pairs are captured in the edges of $H[R_i]_{\beta\nu/8}$. By definition each of the pairs $\{a, b\}$ and $\{c, d\}$ are $\beta$-relevant in $H'$ and thus also in $H$. Observation 2.6 then asserts that for each $z \in N_{H_\delta}(c, d)$ and each $x \in N_{H_\delta}(a, b)$ the pairs $\{a, z\}, \{b, z\}, \{c, x\}, \{d, x\}$ are all $\beta$-relevant in $H$ as well. By (6.4) we conclude that
\[
|N_{H_\delta}(a, b) \cap R|, |N_{H_\delta}(c, d) \cap R|, |N_{H_\delta}(a, z) \cap R|, |N_{H_\delta}(b, z) \cap R|, |N_{H_\delta}(c, w) \cap R|, |N_{H_\delta}(d, w) \cap R| \geq \beta\nu n/4
\]
holds for every $z \in N_{H_\delta}(a, b) \cap R$ and every $x \in N_{H_\delta}(c, d) \cap R$. Then for $n$ sufficiently large
\[
\deg_{H[R_i]}(a, b), \deg_{H[R_i]}(c, d), \deg_{H[R_i]}(a, z), \deg_{H[R_i]}(b, z), \deg_{H[R_i]}(c, x), \deg_{H[R_i]}(d, x) \\
\geq \beta\nu n/4 - 10\gamma \geq \beta\nu n/8
\]
for every $z \in N_{H_\delta}(a, b) \cap R_i$ and every $x \in N_{H_\delta}(c, d) \cap R_i$. This implies that $\{a, b\}$ and $\{c, d\}$ are both $\beta\nu/8$-relevant in $H[R_i]$ as required. Lemma 1.5 then asserts that a 10-path $C_{j+1}$ connects $\{a, b\}$ and $\{c, d\}$ in $H[R_i]$. This establishes the construction of $L_{i+1}$ and thus of $L_\gamma$.

The cycle $L_\gamma$ covers all vertices but those found in $V(H') \setminus V(\mathcal{P})$ and those vertices of $R$ not used for the construction of $L_\gamma$ defined above. The number of these vertices is at most $|V(H') \setminus V(\mathcal{P})| + |R| \leq mn$. A set of vertices of this size can be greedily absorbed into $L_\gamma$ using $A$ rendering a tight Hamilton cycle in $H$.

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**References**


